Flat Electronic Bands and Where to Find Them

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Origin of band flatness and constraints of higher Chern numbers

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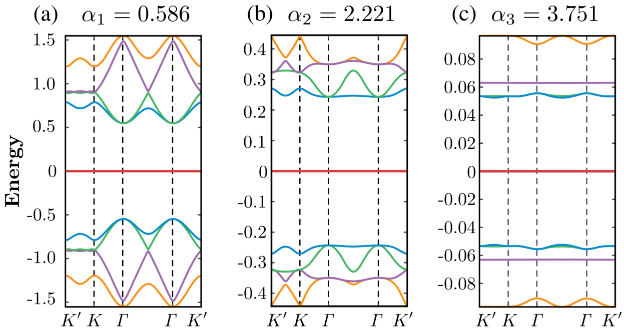
Flat bands provide a natural platform for emergent electronic states beyond Landau paradigm. Among those of particular importance are flat Chern bands, including bands of higher Chern numbers ($C>1$). We introduce a new framework for *band flatness through wave functions*, and classify the existing isolated flat bands in a "periodic table" according to tight binding features and wave function properties. Our flat band categorization encompasses seemingly different classes of flat bands ranging from atomic insulators to perfectly flat Chern bands and Landau Levels. The perfectly flat Chern bands satisfy Berry curvature condition $F_{xy} = \mathrm{Tr}\, \mathcal{G}_{ij}$ which on the tight-binding level is fulfilled only for infinite-range models. Most of the natural Chern bands fall into category of $C = 1$; the complexity of creating higher-C flat bands is beyond the current technology. This is due to the breakdown of the microscopic stability for higher-C flatness, seen atomistically e.g. in the increase of the hopping range bound as $\propto\sqrt{C}a$. Within our new formalism, we indicate strategies for bypassing higher-C constraints and thus dramatically decreasing the implementation complexity.

Inspiration



- Perfectly flat bands in twisted bilayer Graphene

  Tarnopolsky, Kruchkov, Vishwanath, PRL (2019).

Inspiration

- Perfectly flat bands in twisted bilayer Graphene
  Tarnopolsky, Kruchkov, Vishwanath, PRL (2019).
- Haldane argument on the holomorphicity of the higher Landau Levels
  Haldane, Journal Math Phys (2018).
- Qi duality between Landau levels and C=1 Chern bands
  Qi, PRL (2011).
- Jian-Gu-Qi lower bound on the hopping range in perfectly flat bands
  Jian, Gu,Qi, phys status solidi (2013).
- Trescher-Bergholtz construction of higher Chern flat bands in the multilayers
  Trescher, Bergholtz, PRB (2012).

Questions

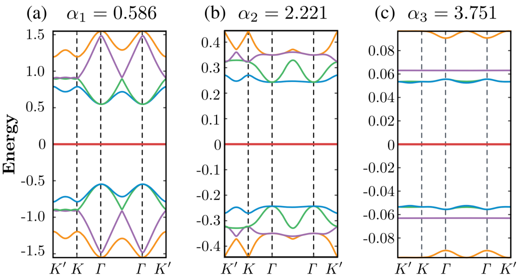
- Is there a common cause of the band flatness?
- Why bringing Landau levels on the lattice (local tight-binding) inevitably broadens the bands?
- Why most of natural flat Chern restricted to C=1?
- Why is it impossible to construct a perfectly flat topological band on the local tight binding? (Chen theorem'2014)
- What is the condition for ideal flat Chern bands expressed through wave functions?
- Can we classify all the known (gapped) perfectly flat bands?

Definition of Band Flatness [1]

- Typically, the band flatness is defined through the *energy definition*

$$\textbf{flatness} = \frac{\text{bandwidth}}{\text{band gap}}$$

- This is a "visual" definition: the band seems to be flat relatively to band scale.



- However, this definition suffers from two problems: 1) it does not have a predictive power; 2) it does not addresses the correlated phenomena (does not guaranty the hierarchy $w \ll U \ll \Delta$).

Definition of the band flatness [2]

- Sometimes, it is more convenient to define band flatness not through energy, but through the energy derivatives, such as Fermi velocity

$$\textbf{flatness} = \frac{\text{Renormalized Fermi velocity}}{\text{Bare Fermi velocity}}$$

- This *qualitative* criterion was used in the first years of twisted bilayer graphene for detecting the magic angles

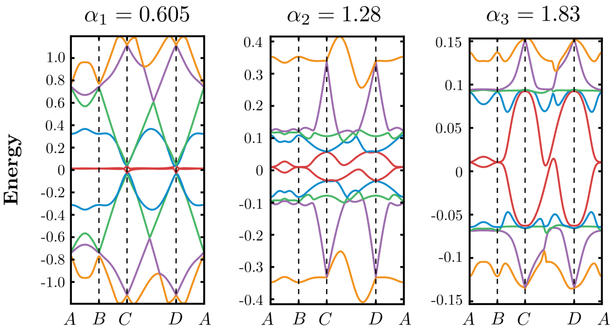
See e.g. Bistritzer, MacDonald, PNAS, (2011).

Tarnopolsky, Kruchkov, Vishwanath, PRL (2019)

Definition of the band flatness [2]

- The problem with this definition is that it often points (incorrectly) on the band flatness even for very dispersive bands

See e.g. Bistritzer, MacDonald, PNAS, (2011).



- Again, this definition suffers from two majors problems: 1) it does not have a predictive power; 2) it does not addresses the correlated phenomena (does not guaranty the hierarchy $U \ll w \ll \Delta$).

Problems with these definitions

- The mentioned above definitions are system-dependent; they visually indicate on the band flatness however may or may not result into the correlated phenomena

- Per se, these definitions are *empirical* and do not contain the information on **band geometry, topology and flatness**, which are fundamentally related.

Demand for the New Definition

- There is a demand for the new definition of the band flatness, which would incorporate all the information of the flat bands themselves.
- More importantly, for the practical reasons, this definition should be compatible with the Wannier formalism and the tigh-binding models; it should work equally good for the trivial bands and for the Chern bands; and it should have a predictive power.

- Such definition **should** be expressed in terms of the *wave functions*, or the *Green's functions*, which contain **all the information about the band geometry, topology and flatness.**

Bird's view on flat bands

- There are so many flat band systems that I simply cannot list them all.

- To systematize them, we require the band to be perfectly flat; if the system can be tuned to the situation when the bands are **perfectly flat**, we call them **fundamentally flat system**.

  TBG flat band: Tarnopolsky, Kruchkov, Vishwanath, PRL (2019)

- We then look for the all classes of fundamentally flat systems.

- Surprisingly, there are just a few classes. Examples: Atomic insulators, fine-tuned artificial lattices (Kagome, Lieb), Landau Levels, Twisted Bilayer Graphene and its descendants.

Common grounds in all the perfectly flat bands

- Surprisingly, we can find a common ground for all the perfectly flat bands: (self-)trapping in the real space

- Harmonic oscillator: self-trapping within the x^2 potential
- Atomic insulator: self-trapping on the atomic sites
- Fine-tuned lattices: self-trapping within the plaquette $\sim a^2$
- Landau levels: self trapping in the area $\sim l_B^2$
- Twisted bilayer graphene: self-traping at the small region (AA) within moiré cell

- The new definition should combine self-trapping criterion and be written in terms of the wave functions

Definition of Band Flatness

- We introduce the new flatness criterion through wave function

$$f = \frac{\sum_{R>\Lambda} |\Psi(R)|^2}{\sum_{R>0} |\Psi(R)|^2} \tag{1}$$

- **Exact Zero** means the band is **perfectly flat**
- This is intuitive for $\delta(R)$ and fine-tuned cases with $\Lambda \sim 2$. Let's show that it works for large Λ.

Proof for large Λ

- **Remark:** $\Lambda \to \infty$ is not physical (non-local Hamiltonian), but it is important for theoretical analysis of 1) band stability; 2) explicit construction of e.g. Landau Levels on the lattice.

See e.g. Kapit, Muller, PRL, (2010).

- Generic argument: we can always construct a perfectly flat band for a non-local Hamiltonian
- Consider a **local** tight-binding Hamiltonian $\mathcal{H} = \sum_{ij}^{\Lambda} t_{ij}^{\alpha\beta} c_{i\,\alpha}^{\dagger} c_{j\,\beta}^{\dagger}$
- Electronic bands $\varepsilon_1(\mathbf{k}), \varepsilon_2(\mathbf{k}) ... \varepsilon_N(\mathbf{k})$
- We can construct a **non-local** Hamiltonian with **perfectly flat band(s)**

$$T_{ij}^{\mathbf{flat}} = \frac{E_0}{N} \sum_{\mathbf{k}} \frac{\mathcal{H}(\mathbf{k})}{\varepsilon_n(k)} e^{i\mathbf{k}(\mathbf{R}_i - \mathbf{R}_j)} \quad \to \quad \mathcal{H}^{\mathbf{flat}} = \sum_{ij}^{\infty} T_{ij}^{\mathbf{flat}} c_{i\,\alpha}^{\dagger} c_{j\,\beta}^{\dagger} \tag{2}$$

- We call this perfectly falt band **generic nontopological nonlocal**.

- The same argument applies to topological bands
- Consider e.g. Haldane model ($\Lambda = 2$)

$$\mathcal{H}_{\text{FDMH}} = \sum_i t_0 \, c_i^\dagger c_i + \sum_{\langle ij \rangle} t_{ij}^{\text{NN}} \, c_i^\dagger c_j^\dagger + \sum_{\langle\langle ij \rangle\rangle} t_{ij}^{\text{NNN}} \, c_i^\dagger c_j^\dagger, \quad t_{ij}^{\text{NNN}} = t' \, e^{i\Phi_{ij}}. \tag{3}$$

- Let's put $\Phi = \pm\pi/2$, the spectrum is particle-hole symmetric, we thus obtain two perfectly flat Chern bands at $E = \pm E_0$ through $T_{ij}^{\textbf{flat}} = \frac{E_0}{N} \sum_{\mathbf{k}} \frac{\mathcal{H}(\mathbf{k})}{\varepsilon_n(k)} e^{i\mathbf{k}(\mathbf{R}_i - \mathbf{R}_j)}$, $\mathcal{H}^{\textbf{flat}} = \sum_{ij}^{\infty} T_{ij}^{\textbf{flat}} \, c_{i\,\alpha}^\dagger c_{j\,\beta}^\dagger$.

  Used for FCI in Neupert et al., PRL, (2011).

- Thus **it is always possible to construct a perfectly flat band for $\Lambda = \infty$, be it topological or not.**
- Since the range Λ of effective tight-binding is related to the wave function's spatial tails, the above statement is consistent with the flatness parameter $f = \frac{\sum_{R>\Lambda} |\Psi(R)|^2}{\sum_{R>0} |\Psi(R)|^2}$.

Flatness of topologically-trivial bands

- We now apply our flatness criterion $f = \frac{\sum_{R>\Lambda} |\Psi(R)|^2}{\sum_{R>0} |\Psi(R)|^2}$ to the topologically trivial bands
- It is more natural to work with Wannier functions; trivial bands are maximally Wannierizable in 2D; for our discussion it is enough to consider Wannierization along one of the 2D axis

$$\mathcal{W}(x - R) \propto \int dk_x \, e^{ik_x(x-R)} \, u_k. \tag{4}$$

- Now, we are interested in the asymptote for large Λ; the claim is if flatness behaves good for large Λ, it is possible to make realistic (nearly flat) band on tight binding $\Lambda \sim 2$.
- A topologically trivial band may have a singularity in the form of a branch vertex in complex momentum $k = k_x + ih_x$, of a generic form $u(k \approx k_*) \simeq u_0[i(k - k_*)]^\alpha$, with $\alpha > -1$.

See e.g. Kohn, PhysRev, (1959).
He, Vanderbilt, PRL, (2001).

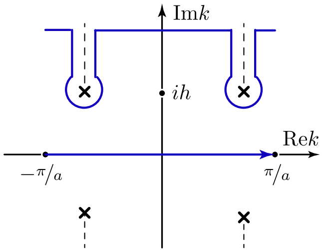
Flatness of topologically-trivial bands

- We need to employ asymptotic analysis here (find $W(x)$ for large x).

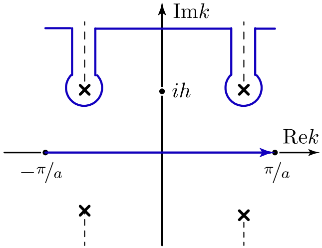
  See e.g. Kohn, PhysRev, (1959).

  He, Vanderbilt, PRL, (2001).

- Let's say $u(k \approx k_*) \simeq u_0[i(k - k_*)]^\alpha$, with $\alpha > -1$. Denote $k_* = k_0 + ih$ as the position of the singularity in the complex plane.



Flatness of topologically-trivial bands



- Asymptotics of Wannier function reduces to integral representation of the Gamma function along a Hankel contour encompassing $k_0 + ih$.

$$\mathcal{W}(x) \simeq 2u_0 \sin(\pi\alpha)\Gamma(1+\alpha)\frac{\exp(-hx)}{x^{1+\alpha}}, \tag{5}$$

- In case of several singularities we take $h = \min[\,\mathrm{Im}\, k_*]$, corresponding to the one closest to the real axis.

Flatness of topologically-trivial bands

- Flatness of topologically trivial bands $f = \frac{\sum_{R>\Lambda} |\Psi(R)|^2}{\sum_{R>0} |\Psi(R)|^2}$
- For this, we need to use analytical estimates of the sums of form $\Sigma_n(\Lambda) = \sum_{x=\Lambda}^{\infty} x^{-n} e^{-x}$. We rewrite this sum as

$$\Sigma_n(\Lambda) = e^{-\Lambda}\, \zeta(\frac{i}{2\pi}, n, \Lambda) \tag{6}$$

where $\zeta(\phi, n, \Lambda) = \sum_{x=0}^{\infty} (x + \Lambda)^{-n} e^{2\pi i \phi x}$ is Lerch zeta function.

See e.g Apostol (1951), Johnson (1974) in Pacific Journal of Mathematics

- Up to $\mathcal{O}(1)$ prefactor Lerch zeta function $\zeta(\frac{i}{2\pi}, n, \Lambda)$ behaves as $1/\Lambda^n$ for $\Lambda \gg 1$, thus we obtain analytical estimate $\Sigma_n(\Lambda) \sim \Lambda^{-n} e^{-\Lambda}$ and $\Sigma_n(1) \sim \mathcal{O}(1/e)$.
- The flatness parameter involves summation of form $\Sigma_n(\Lambda + 1)/\Sigma_n(1) \sim (\Lambda + 1)^{-n} e^{-\Lambda}$, thus

$$f \sim \frac{1}{\Lambda^{2(\alpha+1)}}\, e^{-2h\Lambda a}, \tag{7}$$

- We confirm numerically that this approximation holds.

Flatness of topologically-trivial bands

- We can further simply this criterion

$$f \sim \frac{1}{\Lambda^{2(\alpha+1)}} e^{-2h\Lambda a}, \tag{8}$$

- We are not interested here in the power-law prefactor, and factor of 2 in the exponent. It is safe to rewrite the flatness criterion as

$$f_0 = e^{-h\Lambda a}, \quad \text{for trivial bands.} \tag{9}$$

- The flatness parameter f_0 of (9) sets a fundamental scale for achievable band flatness, and covers three distinguished classes of *perfectly flat* nontopological bands with $f_0 = 0$.

Part I: Flat topologically-trivial bands and where to find them

$$f_0 = e^{-h\Lambda a}, \quad \text{for trivial bands.}$$

- $a \to \infty$, atomic insulator.
- $\Lambda \to \infty$, generic nonlocal, $T_{ij}^{\mathbf{flat}} = \frac{E_0}{N} \sum_{\mathbf{k}} \frac{\mathcal{H}(\mathbf{k})}{\varepsilon_n(k)} e^{i\mathbf{k}(\mathbf{R}_i - \mathbf{R}_j)}$, $\mathcal{H}^{\mathbf{flat}} = \sum_{ij}^{\infty} T_{ij}^{\mathbf{flat}} c_{i\,\alpha}^{\dagger} c_{j\,\beta}^{\dagger}$
- $h \to \infty$, singularity removed to infinity (nonsingular perfectly flat band).

Examples are listed in Ref. Rhim et al, PRB, 2019 .

- Cases of topologically trivial, gapped perfectly flat bands are covered by the three classes above, and constitute the topologically-trivial sector of the flat band classification.
- We do not have evidence of perfectly flat, gapped topologially-trivial bands which do not fit into this classification.

Periodic table of perfectly flat bands

atomic insulator	fine-tuned flat	generic trivial nonlocal	generic topological nonlocal			Landau Level	TBG chiral	
0	$\mathcal{O}(1)$	∞	∞	∞	∞	∞	(∞); $\mathcal{O}(1)$	**Hopping range** Λ
-	none	any	$\sim \pi/a$	$\sim \pi/a$	$\sim \pi/a$	-	- (cancelled; π/λ_M)	**Singularity position** h
0	0	0	$\lvert C\rvert = \frac{1}{2} + \frac{1}{2}$	$\lvert C\rvert = m$	$\lvert C\rvert > 1$	$C = \pm 1$	C=± 1	**Chern number** C
not defined	double-periodic, nonholomorphic	double-periodic, nonholomorphic	double-periodic nonholomorphic	meromorphic non-double-periodic	double-periodic meromorphic	holomorphic quasiperiodic	holomorphic quasiperiodic	**Periodicity in BZ and analyticity**

Flatness of Chern bands

- We now apply the same argument $f = \frac{\sum_{R>\Lambda} |\Psi(R)|^2}{\sum_{R>0} |\Psi(R)|^2}$ to the flatness of the Chern bands

- A theorem, tracing back to Thouless'1984, prevents Wannierizing Chern bands in 2D.

  Thouless, J. Phys. C, (1984) .

- However, it does not prevent Wannierizing a Chern band along one of the 1D directions of a 2D Chern insulator

  Qi, PRL, 2011 .

- For our purposes, localization along 1D is a good indicator of band flatness through $f = \frac{\sum_{R>\Lambda} |\Psi(R)|^2}{\sum_{R>0} |\Psi(R)|^2}$.

- We thus proceed with Wannierizing a Chern band along 1D, and finding its asymptotic behavior.

Chern numbers for a meromorphic flat band

- We start from the construction of higher-C Chern bands on the basis of double periodic meromorphic functions.
- The essential toolbox is built upon implementation of theta functions, Weierstrass and Jacobi functions, and their combinations
- First, we define the Chern numbers for a meromorphic flat band
- We can use connection between the wave function singularities (poles) and the band Chern number $C{=}\int_{\mathrm{BZ}} \frac{d^2\mathbf{k}}{2\pi} F_{xy}$, with $F_{xy}{=}\partial_x A_y{-}\partial_y A_x$, $\mathbf{A}_\mathbf{k}{=}{-}i\langle u_\mathbf{k}|\partial_\mathbf{k} u_\mathbf{k}\rangle$), in the complex plane $z{=}(k_x,k_y)$

$$C = \left[\oint_{\gamma_{\mathrm{BZ}}} + \sum_{z_i^*} \oint_{\gamma_{z_i^*}}\right] \frac{A_{\bar{z}} dz + A_z d\bar{z}}{4\pi} = \sum_{z_i^*} p_i(z_i^*). \tag{10}$$

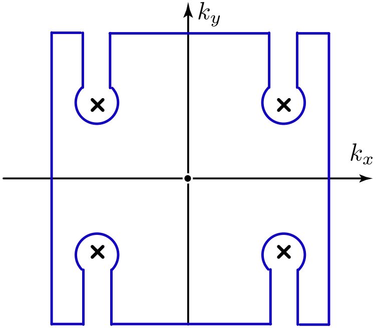
- The Chern number is expressed through the sum of all poles z_i^* in Brilloin zone (BZ), counting their multiplicity $p_i(z_i^*)$

See e.g. Baum, Essays on Topology and Related Topics (1970) .

Chern numbers for a meromorphic flat band

$$C = \left[\oint_{\gamma_{\mathrm{BZ}}} + \sum_{z_i^*} \oint_{\gamma_{z_i^*}} \right] \frac{A_{\bar{z}} dz + A_z d\bar{z}}{4\pi} = \sum_{z_i^*} p_i(z_i^*). \tag{11}$$

- Example for C=4: 4 simple poles, each multiplicity 1.

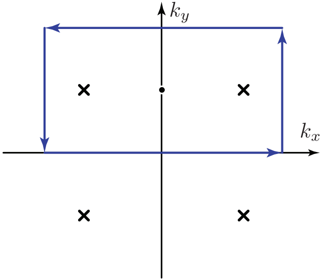


Asymptotics for meromorphic flat bands

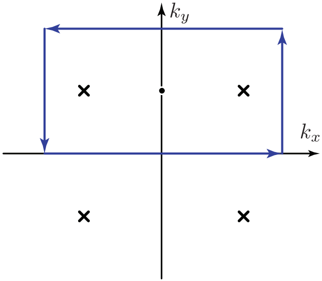
- For asymptotics, it is sufficient to replace the elliptic functions with their principal behavior around poles

$$u(k) \simeq \sum_n \frac{u_0}{[i(k - k_n^*)]^{p_n}} + \text{Regular part.} \tag{12}$$

(here $k = k_x + ik_y$ is in the first BZ).



Asymptotics for meromorphic flat bands



- The main contribution to the Wannier integral is given by the pole (12) of multiplicity $p_n \leq C/2$ closest to the the real axis.
- The residue at the pole is $\operatorname{Res} u(k) = -iu_0 x^{p_n-1} e^{ixk_*}/(p_n-1)!$, with $k_* = k_0 + ih$.
- Using the residue theorem, one obtains the Wannier asymptote

$$\mathcal{W}(x) \simeq \frac{2\pi u_0}{(p_n-1)!} x^{p_n-1} e^{-hx}. \tag{13}$$

Flatness for Chern bands

- To derive the flatness parameter, we need to evaluate sums of form $\Sigma_m(\Lambda) = \sum_{x>\Lambda}^{\infty} x^m e^{-x}$ with m=$2(p_n - 1)$.
- We can rewrite this sum through Lerch zeta function as

$$\Sigma_m(\Lambda) = e^{-\Lambda}\zeta(\frac{i}{2\pi}, -m, \Lambda) \tag{14}$$

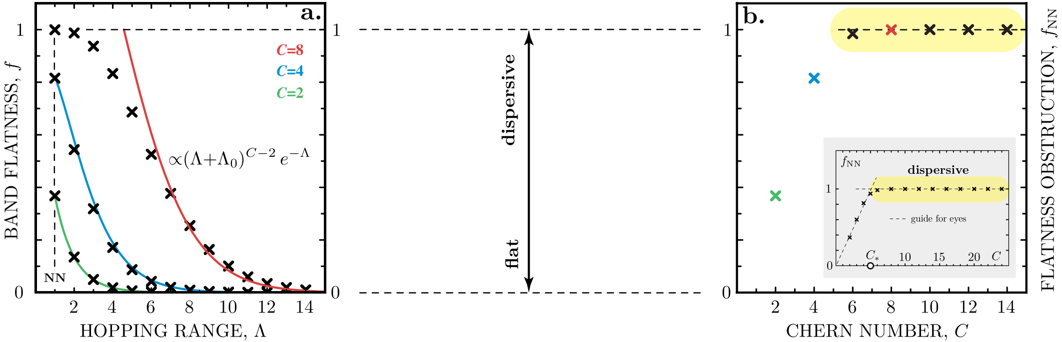
- Up to $\mathcal{O}(1)$ prefactor Lerch zeta function $\zeta(\frac{i}{2\pi}, -m, \Lambda)$ behaves as Λ^m for $\Lambda \gg 1$, thus $\Sigma_m(\Lambda) \sim \Lambda^m e^{-\Lambda}$ and $\Sigma_m(1) \sim \mathcal{O}(1/e)$.
- Restoring dimensional units, we obtain the flatness criterion as $f \sim \Lambda^m e^{-2ha\Lambda}$, where $m = 2(p_n - 1)$, i.e. depends on the nature of the wave function singularities.

Flatness for Chern Bands

- The finite Chern number inevitably leads to constraints on the band flatness
- The high $C{=}N$ can be attained in multiple ways. The simplest way is by having two poles of multiplicity N each. This results into a higher Chern number $C_{2N}{=}2N{\gg}1$ restraining band flatness as $f{\sim}\Lambda^{C_{2N}-2}e^{-2ha\Lambda}$.
- To have a topological band, the wave function singularity must reside inside the BZ.
- This leads to the limitation $h{\leq}\pi/a$ (square lattice), or $ha{\sim}1$ *independently* of a and lattice symmetries.
- The flatness parameter is

$$f \sim \Lambda^{C_{2N}-2}e^{-\Lambda}. \tag{15}$$

Band Flatness Constraints of Higher Chern Numbers



- Perfectly flat Chern bands only for $\Lambda = \infty$
- High Chern number strongly obstructs band flatness
- This could explain why most of natural flat Chern bands limited to C=1.

Chen theorem (2014)

- In a double-periodic system, it is impossible to have perfectly flat Chern bands on the local tight binding.

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **47** (2014) 152001 (12pp) doi:10.1088/1751-8113/47/15/152001

Fast Track Communications

The impossibility of exactly flat non-trivial Chern bands in strictly local periodic tight binding models

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Remarks on Chern band flatness

$$f \sim \Lambda^{C_{2N}-2} e^{-\Lambda}. \tag{16}$$

- Consistent with the Chen theorem (2014).
- Not reducible to the local fine tuning (perfect band flatness)
- Not reducible to the atomic insulator. Chern insulator and atomic insulator belong to different topological classes.

Periodic table of perfectly flat bands

atomic insulator	fine-tuned flat	generic trivial nonlocal	generic topological nonlocal			Landau Level	TBG chiral	
0	$\mathcal{O}(1)$	∞	∞	∞	∞	∞	(∞); $\mathcal{O}(1)$	**Hopping range** Λ
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0	0	0	$\lvert C\rvert = \frac{1}{2} + \frac{1}{2}$	$\lvert C\rvert = m$	$\lvert C\rvert > 1$	$C = \pm 1$	C=±1	**Chern number** C
not defined	double-periodic, nonholomorphic	double-periodic, nonholomorphic	double-periodic nonholomorphic	meromorphic non-double-periodic	double-periodic meromorphic	holomorphic quasiperiodic	holomorphic quasiperiodic	**Periodicity in BZ and analyticity**

Landau Levels and Twisted Bilayer Graphene

- First, we can relax condition of double-periodicity, but still require the flat band state to be a function of $z = k_x + ik_y$.
- In this case the contribution along the BZ boundary, which vanishes due to double-periodicity, may itself contribute to the Chern number.

$$\oint_\gamma \mathbf{A}\, d\mathbf{k} = 2\pi C \tag{17}$$

- This case corresponds to the continuum model of twisted bilayer graphene (TBG), which hosts perfectly flat Chern bands at the magic angle.

  Tarnopolsky, Kruchkov, Vishwanath, PRL, (2019).

- Duality between the perfectly flat Chern bands in TBG and the lowest Landau level

Landau Levels and Twisted Bilayer Graphene

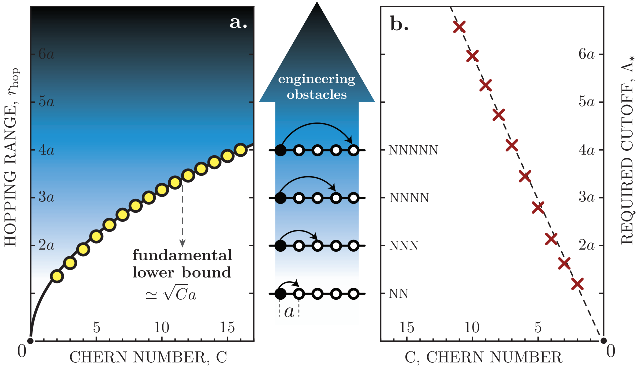
- In both cases we are dealing with effective magnetic fields which produce Berry curvature $F_{xy} \propto l_B^2$, flux 2π through effective Brillouin zone (MBZ) and a perfectly flat band in a certain limit.
- For a generalized Landau level, without loss of generality we can consider asymptote $\mathcal{W}(x) \propto x^n e^{-x^2/2l_B^2}$.
- for Landau Levels the flatness criterion asymptotically reads

$$f_{\mathrm{LL}} \sim \Lambda^{2n-1} e^{-\Lambda^2 a^2/l_B^2}, \tag{18}$$

- Landau levels are perfectly flat only in the nonlocal limit $\Lambda \gg l_B/a \to \infty$.
- Bringing this system on the tight-binding lattice (finite a, finite Λ) inevitably broadens the Landau levels for any finite Λ

Hosftadter, PRB, (1976).
Kapit, Mueller, PRL, (2010).
Dong, Mueller, PRB, (2020).

Topological Constraints on Hopping Range



- Consistent with fundamental bound on hopping range $\sqrt{C}a$ of Jian-Gu-Qi (2013)

Quantum Geometry and Band flatness

- The band topology *and* geometry is described by the ”quantum geometric” tensor

$$\mathfrak{G}_{ij} = \langle \partial_i u_{\mathbf{k}} | \left(1 - |u_{\mathbf{k}}\rangle\langle u_{\mathbf{k}}|\right) | \partial_j u_{\mathbf{k}} \rangle, \tag{19}$$

Roy, PRB, 2014 .
Jackson et al, Nat Comm, 2015

.

- The imaginary part of $\mathfrak{G}$ is responsible for topology, and gives (off-diagonal) Berry curvature $F_{ij} = \mathrm{Im}\mathfrak{G}_{ij}$; the real part $\mathcal{G}_{ij} = \mathrm{Re}\mathfrak{G}_{ij}$ is Fubini-Study metrics and is responsible for the band geometry and its flatness.
- The ideal flat Chern bands satisfy the Berry-geometric condition

$$F_{xy} = \mathrm{Tr}\,\mathcal{G}_{ij} \tag{20}$$

Haldane, PRL, 2011 .
Roy, PRB, 2014 .
Claasen et al., PRL, 2015 .

Quantum Geometry and Band flatness

- The ideal flat Chern bands

$$F_{xy} = \mathrm{Tr}\,\mathcal{G}_{ij} \tag{21}$$

- Holomorphic and meromorphic flat bands automatically satisfy this criterion.
- We can further rewrite

$$F_{xy} = \mathrm{Tr}\,\mathcal{G}_{ij} = \langle u_{\mathbf{k}}|\,|\hat{\mathbf{r}}|^2\,|u_{\mathbf{k}}\rangle, \tag{22}$$

- Integrating (23) over Brillouin zone, one obtains $r_0^2 \propto C$, hence the localization length is $r_0 \sim \sqrt{C}a$.
- For the Chern bands it is impossible to minimize localization length r_0 independently from the hopping range bound $r_{\mathrm{hop}} \sim \sqrt{C}a$; thus the flatness parameter $f = \frac{\sum_{R>\Lambda} |\Psi(R)|^2}{\sum_{R>0} |\Psi(R)|^2}$ cannot be made arbitrary small for any finite Λ, there are always finite tails.

Quantum Geometry and Band flatness

$$F_{xy} = \mathrm{Tr}\, \mathcal{G}_{ij} = \langle u_{\mathbf{k}} | \, |\hat{\mathbf{r}}|^2 \, | u_{\mathbf{k}} \rangle, \tag{23}$$

- Integrating (23) over Brillouin zone, one obtains $r_0^2 \propto C$, hence the localization length is $r_0 \sim \sqrt{C} a$.
- For the Chern bands it is impossible to minimize localization length r_0 independently from the hopping range bound $r_{\mathrm{hop}} \sim \sqrt{C} a$; thus the flatness parameter $f = \frac{\sum_{R>\Lambda} |\Psi(R)|^2}{\sum_{R>0} |\Psi(R)|^2}$ cannot be made arbitrary small for any finite Λ, there are always finite tails.
- Intuitive explanation of Chen theorem (2014).
- Clearly now, the higher Berry fluxes in (23), hence the higher Chern numbers $|C| > 1$, present stronger constraints on electronic band flatness.

Overview: Questions

- Is there a commmon cause of the band flatness?
- Why bringing Landau levels on the lattice (local tight-binding) inevitably broadens the bands?
- Why most of natural flat Chern restricted to C=1?
- Why is it impossible to contruct a perfectly flat topological band on the local tight binding? (Chen theorem'2014)
- What is the condition for ideal flat Chern bands expressed through wave functions?
- Can we classify all the known (gapped) perfectly flat bands?

Periodic table of perfectly flat bands

atomic insulator	fine-tuned flat	generic trivial nonlocal	generic topological nonlocal			Landau Level	TBG chiral	
0	$\mathcal{O}(1)$	∞	∞	∞	∞	∞	(∞); $\mathcal{O}(1)$	**Hopping range Λ**
-	none	any	$\sim \pi/a$	$\sim \pi/a$	$\sim \pi/a$	-	- (cancelled; π/λ_M)	**Singularity position h**
0	0	0	$\lvert C\rvert = \frac{1}{2} + \frac{1}{2}$	$\lvert C\rvert = m$	$\lvert C\rvert > 1$	$C = \pm 1$	C=$\pm$1	**Chern number C**
not defined	double-periodic, nonholomorphic	double-periodic, nonholomorphic	double-periodic nonholomorphic	meromorphic non-double-periodic	double-periodic meromorphic	holomorphic quasiperiodic	holomorphic quasiperiodic	**Periodicity in BZ and analyticity**

Summary

- New criterion for band flatness through wave functions
- Periodic table for perfectly flat bands as building blocks
- Higher-Chern obstructions to band flatness (and ways to bypass them).
- Connection to quantum geometry of flat bands and Fubini-Study metrics.